

ORTHONORMAL SETS WITH NON-NEGATIVE DIRICHLET KERNELS

BY

J. J. PRICE

1. Introduction. With any orthonormal system $\{f_n(s)\}_{n=0}^{\infty}$ of real-valued functions on a measure space S are associated the Dirichlet kernels

$$D_n(s, t) = \sum_{j=0}^{n-1} f_j(s)f_j(t), \quad n \geq 1.$$

If these kernels happen to be non-negative, the Lebesgue functions of the system, i.e. the functions

$$L_n(s) = \int_S |D_n(s, t)| dt, \quad n \geq 1,$$

are much more accessible than otherwise. In particular, if $S = [0, 1]$ and $f_0(s) \equiv 1$, it is immediate that $L_n(s) \equiv 1$ for all n . This is exactly the situation one finds in the case of the classical Haar functions (see [1] or [3]). The uniform boundedness of the functions $L_n(s)$ leads to a proof that the expansion of a continuous function $g(x)$ on $[0, 1]$ in terms of Haar functions converges uniformly to $g(x)$.

It seems natural to ask whether there are other orthonormal sets with the property that $D_n(s, t) \geq 0$ for all n . We shall prove that the answer to this question is essentially negative. It will be shown that only the Haar functions and certain minor modifications of them possess non-negative Dirichlet kernels.

2. Definitions. It will be assumed throughout, except in §5, that μ is a totally finite measure⁽¹⁾ on a space S normalized so that $\mu(S) = 1$.

Using the terminology of [2], a *partition* of S is a finite set P of disjoint subsets of S whose union is S . Given two partitions P_1 and P_2 , $P_1 \geq P_2$ if each element of P_2 is contained in an element of P_1 .

Consider a sequence of partitions of S having the following two properties.

- (i) $P_0 \geq P_1 \geq P_2 \geq \dots$
- (ii) For each $n \geq 0$, $P_n = \{S_{n,1}, S_{n,2}, \dots, S_{n,n+1}\}$ where all sets have positive μ -measure and $S_{0,1} = S$.

These conditions amount to assuming that $P_0 = \{S\}$ and P_{n+1} is obtained from P_n by splitting one of the sets $S_{n,j}$, say for $j = k(n)$, into two subsets of

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⁽¹⁾ See [2] for measure-theoretic definitions.

positive measure. The enumeration can always be arranged so that $S_{n,k(n)}$ splits to form $S_{n+1,1}$ and $S_{n+1,2}$. The sets $S_{n,j}$ ($1 \leq j \leq n+1$, $j \neq k(n)$) are re-labeled and become the sets $S_{n+1,i}$ ($3 \leq i \leq n+2$). To avoid cumbersome subscripts, we shall use k instead of $k(n)$.

With such a sequence of partitions, we associate an orthonormal set $\{f_n(s)\}_{n=0}^\infty$ in $L^2(S, \mu)$ as follows. Put $f_0(s) \equiv 1$. Let $\mu(S_{n,j}) = \mu_{n,j}$ and define for $n \geq 1$

$$(1) \quad f_n(s) = \begin{cases} \left(\frac{\mu_{n,2}}{\mu_{n,1}\mu_{n-1,k}} \right)^{1/2}, & s \in S_{n,1}, \\ - \left(\frac{\mu_{n,1}}{\mu_{n,2}\mu_{n-1,k}} \right)^{1/2}, & s \in S_{n,2}, \\ 0, & \text{otherwise.} \end{cases}$$

A set of functions defined in this way will be called a *Haar system on (S, μ)* . It is easy to verify that a Haar system is an orthonormal set in $L^2(S, \mu)$.

These are obvious generalizations of the orthonormal sets introduced by Haar in [1]. There, S is the unit interval, μ is Lebesgue measure, and the $S_{n,j}$ are sub-intervals. That set in which $\mu_{n,1} = \mu_{n,2}$ ($n \geq 1$) is the one usually called "the Haar functions." (Strictly speaking, the above definition may yield only a subset of the Haar functions. In order to insure completeness, some additional assumption is needed. For instance if μ is nonatomic, one might require that $\lim_{n \rightarrow \infty} \max_j \mu_{n,j} = 0$.)

3. Main theorem. For the sake of simplicity, we are going to avoid "almost everywhere" statements. We shall tacitly identify two functions which differ on a set of measure zero. In this way, for example, an essentially bounded function will be considered bounded.

THEOREM 1. *Let $H = \{f_n(s)\}_{n=0}^\infty$ be a real orthonormal set in $L^2(S, \mu)$ with $f_0(s) \equiv 1$. The Dirichlet kernels associated with H are non-negative if and only if H is a Haar system on (S, μ) .*

The following lemma is needed for the proof of this theorem.

LEMMA. *Let T be a subset of S having measure μ_T , and let c be a positive constant. Define $\mathfrak{F}$ to be the class of functions $f \in L^2(T, \mu)$ satisfying*

$$(A) \quad \int_T f d\mu = 0$$

and

$$(B) \quad f(s)f(t) \geq -c, \quad (s, t) \in T \times T.$$

Then

$$(C) \quad \int_T f^2 d\mu \leq c\mu_T, \quad f \in \mathfrak{F}.$$

The equality in (C) holds for $f^* \in \mathfrak{F}$ if and only if there are two complementary subsets T_1 and T_2 of T with measures μ_1 and μ_2 such that

$$(D) \quad f^*(s) = \begin{cases} \left(\frac{c\mu_2}{\mu_1} \right)^{1/2}, & s \in T_1, \\ - \left(\frac{c\mu_1}{\mu_2} \right)^{1/2}, & s \in T_2. \end{cases}$$

Proof. Given $f \in \mathfrak{F}$, $f = f_1 - f_2$ where $f_1(s) = \max \{f(s), 0\}$ and $f_2(s) = \max \{-f(s), 0\}$. For $i = 1, 2$, let $T_i = \{s | f_i(s) > 0\}$ and let $T_3 = \{s | f(s) = 0\}$. Then the sets T_i are disjoint ($i = 1, 2, 3$) and $\mu_1 + \mu_2 + \mu_3 = \mu_T$ where $\mu_i = \mu(T_i)$.

Because of assumption (A),

$$\int_{T_1} f_1 d\mu = \int_{T_2} f_2 d\mu = I(f).$$

Because of (A) and (B), f is bounded. In fact, if $M_i = \sup_{s \in T} f_i(s)$, ($i = 1, 2$), then $M_1 M_2 \leq c$.

We first maximize $I(f)$. Since $I(f) \leq M_i \mu_i$, ($i = 1, 2$),

$$I(f) \leq \min \{M_1 \mu_1, M_2 \mu_2\} = \min \{M_1 \mu_1, M_2 (\mu_T - \mu_1 - \mu_3)\}.$$

The function of μ_1 on the right above has a unique maximum which occurs when $M_1 \mu_1 = M_2 \mu_2$. One easily obtains

$$(2) \quad I(f) \leq \max_{\mu_1} \min \{M_1 \mu_1, M_2 \mu_2\} = \frac{M_1 M_2}{M_1 + M_2} (\mu_T - \mu_3).$$

Now

$$\int_T f^2 d\mu = \int_{T_1} f_1^2 d\mu + \int_{T_2} f_2^2 d\mu \leq M_1 \int_{T_1} f_1 d\mu + M_2 \int_{T_2} f_2 d\mu = (M_1 + M_2) I(f),$$

and it follows from (2) that

$$(3) \quad \int_T f^2 d\mu \leq M_1 M_2 (\mu_T - \mu_3) \leq c \mu_T.$$

This establishes assertion (C). Furthermore, both equalities hold in (3) if and only if the conditions

$$(4) \quad f(s) \equiv \begin{cases} M_1, & s \in T_1, \\ -M_2, & s \in T_2, \end{cases}$$

$$(5) \quad M_1 \mu_1 = M_2 \mu_2, \quad M_1 M_2 = c,$$

$$(6) \quad \mu_3 = 0,$$

are satisfied simultaneously. (6) shows that T_1 and T_2 must be complementary subsets of T . From (5),

$$M_1 = (M_1 M_2)^{1/2} \left(\frac{M_1}{M_2} \right)^{1/2} = \left(\frac{c \mu_2}{\mu_1} \right)^{1/2};$$

$$M_2 = \left(\frac{c \mu_1}{\mu_2} \right)^{1/2}.$$

Assertion (D) now follows from (4).

4. Proof of Theorem 1. Let $\{f_n(s)\}_{n=0}^\infty$ be a Haar system on (S, μ) . We shall prove by induction that for $n \geq 0$,

$$(7) \quad D_{n+1}(s, t) = \begin{cases} \frac{1}{\mu_{n,j}}, & (s, t) \in S_{n,j} \times S_{n,j}, (1 \leq j \leq n+1), \\ 0, & \text{otherwise.} \end{cases}$$

This is true for $n=0$ since $f_0(s) \equiv 1$, $D_1(s, t) \equiv 1$, and $\mu_{0,1} = \mu(S) = 1$. In general

$$(8) \quad D_{n+2}(s, t) = D_{n+1}(s, t) + f_{n+1}(s)f_{n+1}(t).$$

By definition $f_{n+1}(s) \equiv 0$ outside $S_{n,k}$. Hence $D_{n+2}(s, t) = D_{n+1}(s, t)$ outside $S_{n,k}^2 = S_{n,k} \times S_{n,k}$. From the induction hypothesis (7) this means

$$(9) \quad \text{if } (s, t) \in S_{n,k}^2, D_{n+2}(s, t) = \begin{cases} \frac{1}{\mu_{n+1,j}}, & (s, t) \in S_{n+1,j}^2, (3 \leq j \leq n+2), \\ 0, & \text{otherwise.} \end{cases}$$

Now consider $(s, t) \in S_{n,k}^2 = (S_{n+1,1} \cup S_{n+1,2})^2$. From the definition (1),

$$(10) \quad f_{n+1}(s)f_{n+1}(t) = \begin{cases} \frac{\mu_{n+1,2}}{\mu_{n+1,1}\mu_{n,k}}, & (s, t) \in S_{n+1,1}^2, \\ \frac{\mu_{n+1,1}}{\mu_{n+1,2}\mu_{n,k}}, & (s, t) \in S_{n+1,2}^2, \\ \frac{1}{\mu_{n,k}}, & \text{otherwise.} \end{cases}$$

But by the induction hypothesis and (8),

$$(11) \quad D_{n+2}(s, t) = f_{n+1}(s)f_{n+1}(t) + \frac{1}{\mu_{n,k}}$$

when $(s, t) \in S_{n,k}^2$. From (10) and (11), we easily find that

$$(12) \quad \text{if } (s, t) \in S_{n,k}^2, D_{n+2}(s, t) = \begin{cases} \frac{1}{\mu_{n+1,1}}, & (s, t) \in S_{n+1,1}^2, \\ \frac{1}{\mu_{n+1,2}}, & (s, t) \in S_{n+1,2}^2, \\ 0, & \text{otherwise.} \end{cases}$$

Statements (9) and (12) together are equivalent to equation (7) with $n+1$ replaced by $n+2$, completing the induction.

Suppose now that $H = \{f_n(s)\}_{n=0}^\infty$ is an orthonormal set in $L^2(S, \mu)$, $f_0(s) \equiv 1$, and the associated Dirichlet kernels are non-negative. We shall prove by induction that for each $n \geq 0$ there is a sequence of partitions $P_0 \geq P_1 \geq P_2 \geq \cdots \geq P_n$ satisfying (i) and (ii), that $f_0(s), f_1(s), \cdots, f_n(s)$ are the associated Haar functions, and that (7) holds.

For $n=0$, these assertions are obvious. Assume they are true for n . $D_{n+2}(s, t) \geq 0$. Therefore, from (8) and the induction hypothesis

$$(13) \quad f_{n+1}(s)f_{n+1}(t) \geq -D_{n+1}(s, t) = \begin{cases} -\frac{1}{\mu_{n,j}}, & (s, t) \in S_{n,j}^2, (1 \leq j \leq n+1), \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $f_{n+1}(s) \equiv 0$ on all but one of the sets $S_{n,j}$ ($1 \leq j \leq n+1$). Since $f_{n+1}(s) \not\equiv 0$ on S , we can find a set $S_{n,k}$ on which $f_{n+1}(s) \not\equiv 0$. Suppose $f_{n+1}(s) \not\equiv 0$ outside $S_{n,k}$. Then there are points $s \in S_{n,k}$ and points $t \notin S_{n,k}$ where the function is nonzero. For any pair (s, t) of such points, we have from (13) that $f_{n+1}(s)f_{n+1}(t) > 0$. It follows that either $f_{n+1}(s) \geq 0$ or $f_{n+1}(s) \leq 0$ for all $s \in S$. But this is impossible since

$$\int_S f_{n+1} d\mu = \int_S f_0 f_{n+1} d\mu = 0.$$

Therefore, $f_{n+1}(s) \equiv 0$ outside $S_{n,k}$.

Consider now the restriction of f_{n+1} to the subset $S_{n,k}$. The following three statements hold.

$$(A) \quad \int_{S_{n,k}} f_{n+1} d\mu = 0.$$

$$(B) \quad f_{n+1}(s)f_{n+1}(t) \geq -\frac{1}{\mu_{n,k}}.$$

$$(C) \quad \int_{S_{n,k}} f_{n+1}^2 d\mu = 1.$$

(A) and (C) express orthonormality; (B) is a restatement of part of (13). Now (A) and (B) are precisely the conditions of the lemma with $\mu_T = \mu_{n,k}$ and $c = 1/\mu_{n,k}$. We conclude

$$\int_{S_{n,k}} f_{n+1}^2 d\mu \leq c\mu T = 1.$$

But now (C) shows that f_{n+1} is maximal in the sense of the lemma. Therefore, there exist two complementary subsets $S_{n+1,1}$ and $S_{n+1,2}$ of $S_{n,k}$ having measures $\mu_{n+1,1}$ and $\mu_{n+1,2}$ such that

$$f_{n+1}(s) = \begin{cases} \left(\frac{\mu_{n+1,2}}{\mu_{n+1,1}\mu_{n,k}} \right)^{1/2}, & s \in S_{n+1,1}, \\ -\left(\frac{\mu_{n+1,1}}{\mu_{n+1,2}\mu_{n,k}} \right)^{1/2}, & s \in S_{n+1,2}, \\ 0, & \text{otherwise.} \end{cases}$$

For $3 \leq j \leq n+2$, define sets $S_{n+1,j}$ to be the sets $S_{n,i}$ ($1 \leq i \leq n+1$, $i \neq k$) taken in any order. Clearly $P_{n+1} = \{S_{n+1,j}\}_{j=1}^{n+2}$ is a partition of (S, μ) , $P_0 \geq P_1 \geq \dots \geq P_n \geq P_{n+1}$ is an extension of the sequence of partitions already defined, $f_{n+1}(s)$ is the Haar function associated with P_{n+1} , and (7) follows for $n+2$ exactly as in the first part of the proof. This completes the proof by induction that H is a Haar system on (S, μ) .

5. An extension of Theorem 1. We now drop the assumption that (S, μ) is totally finite and weaken the restriction that $f_0(s) \equiv 1$.

THEOREM 2. Let $\{f_n(s)\}_{n=0}^{\infty}$ be a real orthonormal set in $L^2(S, \mu)$. The associated Dirichlet kernels are non-negative if and only if

(A) $f_0(s)$ is either non-negative or nonpositive and

(B) for each $n \geq 0$, $f_n(s) = f_0(s)\phi_n(s)$ where $\{\phi_n(s)\}_{n=0}^{\infty}$ is a Haar system on (S, ν) and $d\nu = f_0^2(s)d\mu$.

Proof. To prove sufficiency of these conditions, note that (S, ν) is totally finite and $\nu(S) = 1$. Therefore, by Theorem 1,

$$\sum_{j=0}^{n-1} \phi_j(s)\phi_j(t) \geq 0.$$

But

$$D_n(s, t) = \sum_{j=0}^{n-1} f_j(s)f_j(t) = f_0(s)f_0(t) \sum_{j=0}^{n-1} \phi_j(s)\phi_j(t).$$

Since $f_0(s)f_0(t) \geq 0$ from condition (A), $D_n(s, t) \geq 0$.

Next, we show necessity. Since $f_0(s)f_0(t) = D_1(s, t) \geq 0$ condition (A) holds.

Let $Z = \{s | f_0(s) = 0\}$. We shall prove by induction that $f_n(s) \equiv 0$ on Z for $n \geq 0$. Suppose this is true for $0, 1, 2, \dots, n$.

$$0 \leq D_{n+2}(s, t) = f_{n+1}(s)f_{n+1}(t) + \sum_{j=0}^n f_j(s)f_j(t).$$

If $s \in Z$, the second term on the right vanishes because of the induction hypothesis, so that $f_{n+1}(s)f_{n+1}(t) \geq 0$. Suppose $f_{n+1}(s) \neq 0$ on Z . Let $s \in Z$ and $t \in S$ be points where the function is nonzero. Then $f_{n+1}(s)f_{n+1}(t) > 0$. It follows that either $f_{n+1}(s) \geq 0$ or $f_{n+1}(s) \leq 0$ for all $s \in S$. But this is impossible because then either $f_0(s)f_{n+1}(s) \geq 0$ or $f_0(s)f_{n+1}(s) \leq 0$ for all s contradicting the fact that

$$\int_S f_0 f_{n+1} d\mu = 0.$$

Therefore $f_{n+1}(s) = 0$ on Z .

We may now define

$$\phi_n(s) = \begin{cases} \frac{f_n(s)}{f_0(s)}, & s \in Z, \\ 0, & s \in S. \end{cases}$$

Let $d\nu = f_0^2(s)d\mu$. Then the set $\{\phi_n(s)\}_{n=0}^\infty$ is an orthonormal set in $L^2(S, \nu)$ for

$$\int_S \phi_j \phi_k d\nu = \int_{S-Z} \frac{f_j f_k}{f_0^2} f_0^2 d\mu = \int_S f_j f_k d\mu = \delta_{jk}$$

and

$$\int_S \phi_j^2 d\nu = \int_{S-Z} \left(\frac{f_j}{f_0} \right)^2 f_0^2 d\mu = \int_S f_j^2 d\mu = 1.$$

Furthermore, the Dirichlet kernels associated with this set are non-negative since

$$\sum_{j=0}^{n-1} \phi_j(s)\phi_j(t) = \begin{cases} 0, & s \in Z \text{ or } t \in Z, \\ \frac{1}{f_0(s)f_0(t)} \sum_{j=1}^{n-1} f_j(s)f_j(t) = \frac{D_n(s, t)}{D_1(s, t)} \geq 0, & \text{otherwise.} \end{cases}$$

By Theorem 1, $\{\phi_n(s)\}_{n=0}^\infty$ is a Haar system on (S, ν) . Since $f_n(s) = f_0(s)\phi_n(s)$ for $n \geq 0$, the theorem is proved.

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CORNELL UNIVERSITY,
ITHACA, NEW YORK